The diameter of long-range percolation clusters on finite cycles

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Abstract

Bounds for the diameter and expansion of the graphs created by long-range percolation on the cycle $\mathbb{Z}/N\mathbb{Z}$, are given.

1 Introduction

Model the metric of the world at the year 1900 by some graph, e.g. $\mathbb{Z}^2$ or $\mathbb{Z}^3$, equipped with the graph metric. The introduction of fast communication and transportation starting with phones, cars, airplanes and finally (for now) the Internet, decreases distances. This can be modeled by adding new edges between far away vertices in one way or another. A natural way to do that, is long-range percolation. In (Bernoulli) long-range percolation, a countable set of vertices V is given, equipped with a distance function d, on the set. Now to get a random graph with V as its vertices set, attach an edge $e_{v,u}$ between $v, u \in V$ with probability $p_{d(v,u)}$, determined only by the distance between v and u, independently of all other pair of vertices. Long-range percolation on $\mathbb{Z}$ was introduced and studied in [15], [13] and [2]. These papers mainly studied when an infinite cluster exists, whether it is unique (yes [9]) and the type of phase transitions that occurs, see also [12]. Not much attention was given to the geometry and structure of the infinite cluster, once it exists. In [5] and [3] the random walk on and volume growth of long-range percolation clusters on $\mathbb{Z}$ and $\mathbb{Z}^2$ were studied. We then observed that even on finite graphs long-range percolation might be of interest. In particular in trying to study the world wide web. The spatial structure of the world still manifests itself in the web structure. These ideas are not new, see [16] [17], and appear occasionally under the name "small world".

The attachment probabilities we will consider will have polynomial decay in the distance, i.e. $p_{d(v,u)} \sim \beta d^{-s}$. In the next section the model is defined, in Section 3 we consider the diameter of the clusters, Section 4 contains the formulation of a sharp result (quoted from [4]) regarding the diameter of the cluster for $s < 1$. In Section 5 we discuss expansion properties of the cluster. We end with some less formal concluding remarks.

Similar models have been discussed in the computer science and physics literature. In [11] Kleinberg has studied the properties of a quite similar model. His interest regarded a two-dimensional

fixed degree model for which the probability of a long range edge of length d to be open is proportional to d^{-s}. His focus was on constructing good routing algorithms that rely only on local information.

In [7], the behavior of random walk on models of this sort is studied. Jespersen and Blumen ([7]) study the return probabilities of the random walker in a slightly different model. Their study, as well as [5], reveals a phase transition at $s = 2$. The same phase transition (as well as one at $s = 1$) shows also as a result of our study. A very interesting continuation of this work was recently written by Coppersmith, Gamarnik and Sviridenko ([8]).

2   The model

The model we discuss is the finite long-range percolation model with polynomial decay. Let N be a positive integer, let $s, \beta > 0$, and consider the following random graph:
The vertices are the elements of the cycle $\mathbb{Z}/N\mathbb{Z}$. Define $\rho(x, y) = \min(|x - y|, N - |x - y|)$.
Determine the edges in the graph as follows:
If $\rho(x, y) = 1$, then x and y will be attached to each other. Otherwise, if $x \neq y$, then x and y will be attached with probability $1 - \exp(-\beta\rho(x, y)^{-s})$. The different edges are all independent of each other. The probability of an edge between to (distant enough) vertices is very close to $\beta\rho(x, y)^{-s}$, and this, as well as independence, are the two important features of the presented distribution.

We call the graph created this way $G_{s,\beta}(N)$.

Other interesting models could be high dimensional models, models with different decay rate (exponential or other), or models with dependencies, like the Random Cluster Model or models of the type discussed in [11].

3   The diameter

In this section, we fix s and β, and estimate the diameter of the graph. For N, s and β, let $\mathcal{D}(N) = \mathcal{D}_{s,\beta}(N)$ be the diameter of $G_{s,\beta}(N)$. The main results are

Theorem 3.1. *(A) If $s > 2$ then there is $C = C_{s,\beta}$ s.t.*

$$\lim_{N \to \infty} \mathbf{P}(\mathcal{D}(N) < CN) = 0.$$

Moreover, there exists $0 < \eta \leq 1$ s.t. $\mathcal{D}(N)/N \to \eta$ in distribution.
(B) If $s < 1$ then there is constant $C = C_{s,\beta}$ s.t.

$$\lim_{N \to \infty} \mathbf{P}(\mathcal{D}(N) > C) = 0$$

.
(C) If $1 < s < 2$ then there is constant $\delta = \delta_{s,\beta}$ s.t.

$$\lim_{N \to \infty} \mathbf{P}(\mathcal{D}(N) > \log(N)^{\delta}) = 0$$

.

(D) if $1 < s < 2$, then there is a constant $C = C_{s,\beta}$ s.t.

$$\lim_{N \to \infty} \mathbf{P}(\mathcal{D}(N) < C \log(N)) = 0$$

.

At a previous version of this paper, we have conjectured:

Conjecture 3.2. *(A) If $s = 2$, then the diameter's order of magnitude is N^δ, where δ is a function of β.*
(B) If $s = 1$, then the diameter's order of magnitude is $\log(N)$.
(C) If $1 < s < 2$ then the diameter is $\theta(\log(N)^\gamma)$ where $\gamma > 1$ is a function of s.

Recently, Coppersmith, Gamarnik and Sviridenco ([8]) have proved that for $s = 1$, the diameter is of order of magnitude $\log(N)/\log\log(N)$. This contradicts part (A) of conjecture 3.2. At the same paper they also proved part (B) of Conjecture 3.2 for $\beta < 1$. For all other values of β, they have proved an upper bound for the diameter of order of magnitude N^δ where $\delta < 1$ and its value depends on β. Further, Biskup ([6]) has recently announced that when $1 < s < 2$ the diameter is $\log(N)^{\log_2(2/s)+o(1)}$. This proves part (C) of conjecture 3.2.

In view of these results, we now believe:

Conjecture 3.3. *(A) If $s = 2$, then the diameter's order of magnitude is N^δ, where δ is a function of β.*
(B) If $1 < s < 2$ then the diameter is $\theta(\log(N)^\gamma)$ where $\gamma > 1$ is a function of s.

In [3] it is shown that for $s = 2$ and any β the diameter is no less than $N^{1/\log\log N}$.

Proof of Theorem 3.1. (A) Assume first that the model is the line $[0, N-1]$ with $\rho(x, y) = |x - y|$ and not the circle. Now, for given x_0, the probability that there is no edge between any $x < x_0$ and any $y > x_0$ is

$$
\begin{aligned}
\prod_{0 \le x < x_0, N > y > x_0} e^{-\beta(y-x)^{-s}} &= e^{-\beta \sum_{0 \le x < x_0, N > y > x_0} (y-x)^{-s}} \\
&\ge e^{-\beta \sum_{x < x_0, y > x_0} (y-x)^{-s}} \\
&= e^{-\beta \sum_{k=1}^{\infty} k^{1-s}} \\
&= \psi(s, \beta) > 0.
\end{aligned}
$$

Define a *cut* to be a vertex with this property. Take C s.t. $6C = \frac{1}{2}\psi(s, \beta)$. So, by the ergodic theorem, if N is large enough then, with probability as high as we like, there are at least $6CN = \frac{1}{2}\psi(s, \beta)$ cuts, and therefore the diameter is, with the same probability, at least $6CN$. Returning to the cycle, we can divide it into two lines, of length $\frac{1}{2}N$ each. Each of these halves is of diameter at least $3CN$. The same kind of calculation yields that, with high probability, the edges between the two halves of the cycle don't reach the middle third of each of the lines of the vertices, and

3

therefore the diameter stays above CN. In order to prove that the limit of $\mathcal{D}(N)/N$ exists, we do the following: First, consider long-range percolation on $\mathbb{Z}$. $\mathcal{D}'(N)$ will be the diameter of the long-range percolation restricted to $[0, N]$. By the sub-additive ergodic theorem, $D'(N)/N \to \eta$ a.s. for some $\eta > 0$. In order to prove the convergence for the diameter of the cycle, we divide the cycle $\mathbb{Z}/N\mathbb{Z}$ into two intervals I_1 and I_2 of length $N/2$. The diameter of each half is (with very high probability) approximately $\eta N/2$. The longest connection between the two halves is of length $o(N)$, so there are cut points $x_1, x_2 \in I_1$ and $x_3, x_4 \in I_2$ s.t. $\rho(x_1, x_2) = N/2 - o(N)$ and $\rho(x_3, x_4) = N/2 - o(N)$, there are no edges between the arcs $[x_1, x_2]$ and $[x_3, x_4]$, and, by the strong Markov property, the diameter of each of these arcs is $\eta N/2 - o(N)$. So, we are done. It is of interest to study the fluctuations of $D(N)/N$ from η.

(B) The graph dominates the $G(n, p)$ random graph with edge probability βN^{-s}. It is known (see, e.g. [10]) that there exists a constant C s.t.

$$\lim_{n \to \infty} \mathbf{P}(\mathcal{D}(G(n, \beta n^{-s})) < C) = 1 \tag{1}$$

Since the diameter is a decreasing function (w.r.t the standard partial order), (1) applies also for our model with $s < 1$.

Actually, in this case we can even say more: The infinite graph whose vertices are the integers, s.t. every two vertices are attached with probability $1 - \exp(-\beta G|x - y|^s)$ has, a.s., a finite diameter, see next section.

(C) Here we use an argument in the spirit of Newman and Schulman's renormalization (see [13]): Again, assume that the model is a line instead of a circle. This assumption creates a measure which is dominated by the original one, and therefore it suffices to prove the result for the line. Take

$$C_i = e^{\alpha^i},$$

where $\alpha > 1$ is s.t.

$$s < (2 - s) \sum_{i=1}^{\infty} \alpha^{-i}$$

Let k_0 be a large number, and define γ to be

$$\gamma = (2 - s) \sum_{i=1}^{k_0} \alpha^{-i} - s.$$

Taking α small enough and k_0 large enough, we can get that

$$e^{\gamma} > \alpha \tag{2}$$

Now, take

$$N_k = \prod_{i=1}^{k} C_k.$$

We divide the interval of length N_k into C_k intervals of length N_{k-1}. Each of these, we divide into C_{k-1} intervals of length C_{k-2}, and so on. This structure has a lot in common with the one

used in [13] for proving the existence of the infinite cluster. We use the following terminology: The N_l/N_k intervals of length N_k obtained by this division from N_l, are called *components of degree k*. The components of degree $k-1$ inside such a component are *sub-components*. two intervals (or component) I and J are said to be *attached* to each other, if there exists a bond between a point in I and a point in J.

It is enough to show that for some constant D,

$$\lim_{k\to\infty} \mathbf{P}(\mathcal{D}(N_k) \leq D^k) = 1, \tag{3}$$

because

$$D^k = \left(\alpha^k\right)^\delta < \log(N_k)^\delta$$

for $\delta = \frac{\log D}{\log \alpha}$, and if $N_k < n < N_{k+1}$, then we can bound $\mathcal{D}(n)$ by $\mathcal{D}(N_{k+1})$. This will be enough because for some constant κ, we have $\log(N_k) \leq \kappa \log(n)$.

We now prove (3): Take some $\epsilon > 0$. We will show that for l large enough, $\mathbf{P}(\mathcal{D}(N_l) \leq D^l) > 1 - \epsilon$. Define $\beta_k = \frac{1}{2}\beta N_k^{2-s}$. Then the probability that two intervals of length N_k of distance lN_k from each other have an edge between them is at least $1 - \exp(-\beta_k l^{-s})$.

Take $k_1 > k_0$ so large that for every $k > k_1$, we have $\frac{1}{6}\beta e^{\gamma \alpha^k} > 2\alpha^k$, and so that $e^{-k_1} < \epsilon/2$. For every $k > k_1$, consider the line of length N_k. Divide it into C_k components of size N_{k-1}. The probability that not all of the C_k components are attached to each other is bounded by

$$
\begin{aligned}
\binom{C_k}{2} e^{\left(-\beta_{k-1} C_k^{-s}\right)} &\leq C_k^2 e^{-\beta_{k-1} C_k^{-s}} \\
&= C_k^2 \exp\left(-\frac{1}{2}\beta e^{-s\alpha^k} e^{(2-s)\sum_{i=1}^{k-1} \alpha^i}\right) \\
&\leq \exp\left(2\alpha^k - \frac{1}{2}\beta e^{\gamma \alpha^k}\right) \\
&\leq \exp\left(-\frac{1}{3}\beta e^{\gamma \alpha^k}\right)
\end{aligned}
$$

Now, take l s.t. $\psi \log l > k_1$ where ψ is s.t. $\psi \log \alpha = 1$. Consider the following event, denoted by ν: for every $\psi \log l \leq k \leq l$, and for every component of degree k, all of its sub-components of degree $k-1$ are attached to each other.

Given ν, the diameter is no more than

$$2^l N_{\psi \log l} < 2^l \left(e^{(1/(1-\alpha^{-1}))\alpha^{\psi \log l}}\right) < D^l$$

for $D = 2\exp(1/(1-\alpha^{-1}))$. Therefore, we want to estimate the probability of ν:

Take k s.t. $\psi \log l \leq k \leq l$. The probability that there exist a component of degree k, s.t. not all of its sub-components of degree $k-1$ are attached to each other could be bounded by the number

of components of degree k times the probability for this event at each of them, i.e. by

$$
\begin{aligned}
N_k \cdot \exp\left(-\frac{1}{3}\beta e^{\gamma\alpha^k}\right) &\leq N_l \cdot \exp\left(-\frac{1}{3}\beta e^{\gamma\alpha^k}\right) \\
&\leq \exp\left((1/(1-\alpha^{-1}))\alpha^l - \frac{1}{3}\beta e^{\gamma\alpha^{\psi\log l}}\right) \\
&= \exp\left((1/(1-\alpha^{-1}))\alpha^l - \frac{1}{3}\beta e^{\gamma l}\right) \\
&\leq \exp\left(-\frac{1}{4}\beta e^{\gamma l}\right)
\end{aligned}
$$

by (2) for large enough l.

Therefore, the probability of ν is at least

$$
1 - l\exp\left(-\frac{1}{4}\beta e^{\gamma l}\right) > 1 - \epsilon
$$

for l large enough.

So,

$$
\mathcal{D}(N) = O\left(\log(N)^{\frac{\log(2\exp((1-\frac{2s}{2+s})^{-1}))}{\log(2+s)-\log(2s)}+\epsilon}\right)
$$

for every ϵ.

(D) When $s > 1$, the expected value of the number of vertices attached to a certain vertex is finite. Therefore, the graph's growth rate is bounded by the growth rate of a Galton-Watson tree. Thus, its growth rate is (bounded by) exponential, and so the diameter cannot be smaller than a logarithm of the number of vertices (N, in this case). □

4 More on $s < 1$

In this section we will report on a theorem from [4] for long-range percolation, that deals with the case $s < 1$.

Denote by B_N^d the standard d-dimensional square lattice restricted to the $N \times N .... \times N$ box. The diameter of B_N^d, in the graph metric, denoted $d(\, , \,)$, is dN. We will add random edges to B_N^d as follows. Fix some $s \in (0, d)$, and for any two vertices $v, u \in B_N^d$, add an edge between v, u with probability $d(v, u)^{-s}$, independently from all other edges. Denote by $D(B_N^d)$ the random diameter of the box once the new edges were added.

Theorem 4.1. *([4])*

$$
\lim_{N\to\infty} D(B_N^d) = \lceil \frac{d}{d-s} \rceil \ a.s.
$$

Remarks.

6

- This theorem was proved in [4] as a corollary of the general theory of stochastic dimension that was developed for the study of uniform spanning forest. A random relation $\mathbf{R} \subset \mathbb{Z}^{\mathbf{d}} \times \mathbb{Z}^{\mathbf{d}}$ is said to have *stochastic dimension s*, if there is some constant $c > 0$ such that for all $x \neq y$ in $\mathbb{Z}^d$,

$$c^{-1}|x - y|^{s-d} < \mathbf{P}[x\mathbf{R}y] < \mathbf{c}|\mathbf{x} - \mathbf{y}|^{\mathbf{s-d}},$$

  and certain correlation inequalities hold. The results regarding stochastic dimension are formulated and proven in this generality, to allow application in several contexts. One application is the above. Another application is the following:

  For every $v \in B_N^d$, let S^v be a simple random walk starting from v, with $\{S^v\}_{v \in B_N^d}$ independent. Let v knows u be the relation $\exists n \in \mathbf{N}$ $S^v(n) = S^u(n)$. Then "knows" has stochastic dimension 2. The "know" diameter (i.e. the diameter of the graph in which there is an edge between v and u whenever v knows u) of B_N^d will be $\lceil \frac{d}{d-2} \rceil$ a.s.

- If indeed we are all six handshakes away from any other person on the planet, then assuming $d = 2$, the "real world handshakes exponent" s, might be around $1 - 2/7$.

5 Cheeger's Constant

We would like to explore the Cheeger constant of these graphs. First, recall its definition. For a set of vertices A in a finite graph, let ∂A be the set of edges $\{e = (v, u), v \in A, u \in A^c\}$. A geometric tool which is used in order to bound relaxation times of random walk on graphs is the Cheeger constant $\mathcal{C}(G)$, see [1].

$$\mathcal{C}(G) = \inf_A \frac{\partial A}{|A|}, \tag{4}$$

where the infimum is taken over all non empty set of vertices A, with $|A| \leq \frac{|G|}{2}$.

Again, fix s and β, and denote by $\mathcal{C}(N) = \mathcal{C}_{s,\beta}(N)$ the Cheeger constant of the graph $G_{s,\beta}(N)$ when $s > 2$, the Cheeger constant is $\theta(\frac{1}{N})$. That is proven the same way as the fact that the diameter is linear in N. However, for $1 < s \leq 2$, the Cheeger constant exhibits an interesting behavior:

Theorem 5.1. *(A) If $1 < s < 2$, then there is a constant $\alpha = \alpha(s)$ s.t.*

$$\lim_{N \to \infty} \mathbf{P}\left(\mathcal{C}(N) > N^{-\alpha}\right) = 0.$$

(B) If $s = 2$, then for every ϵ, there C s.t.

$$\limsup_{N \to \infty} \mathbf{P}\left(\mathcal{C}(N) > \frac{C \log N}{N}\right) < \epsilon.$$

Part (B) is proved as lemma 3.4 of [5].

Proof of (A). Divide the circle into two arcs, A and B, of length $\frac{1}{2}N$ each. The expected value of the number of edges connecting the two halfs is:

$$
\begin{aligned}
2 \; &+ \sum_{i \in A, j \in B} (1 - \exp(-\beta \rho(i,j)^{-s})) \\
&\leq \; 2 + 2\sum_{k=2}^{\frac{1}{2}N} k(1 - \exp(-\beta k^{-s})) \\
&\leq \; 2 + C\sum_{k=2}^{\frac{1}{2}N} k^{1-s} \leq 2 + 2C\beta \sum_{k=2}^{\infty} k^{1-s} \\
&\leq \; 2 + \frac{C}{2-s}\left(\frac{N}{2}+1\right)^{2-s}
\end{aligned}
$$

For some constant C. So, the expected value of the size of the boundary of A divided by the size of A is bounded by DN^{1-s} for some constant D, and this, using the Markov inequality, gives the desired result for every $\alpha < s - 1$. □

As a simple corollary of Theorem 6.1, we get the following lower bound for the mixing time of a random walk on $G_{s,\beta}(N)$. (See [1] for background on mixing)

Corollary 5.2. *Denote by $\tau(G)$ the mixing time of the simple random walk on a graph G. If $1 < s < 2$ then for every $\delta < s$, we have*

$$
\lim_{n \to \infty} \mathbf{P}(\tau(G_{s,\beta}(N)) < N^{\delta}) = 0
$$

By theorem 6.1 we see that the size of the smallest cut in the case $1 < s < 2$ reminds that of a cube in Z^d. However, the cut structure is different. While in a cube of length $N^{1/d}$ (and, therefore, N vertices) in Z^d one can find a sequence of length $\theta(N^{1/d})$ of nested cuts of size $O(N^{\frac{d-1}{d}})$ each, in $G_{s,\beta}(N)$ the length of a sequence of nested cuts is bounded by the diameter which is no more than poly-logarithmic in N.

6   Concluding Remarks

6.1   Discussion

The geometry of the $1 \leq s < 2$ clusters described here is different from the geometry of other natural graphs - In the $s > 1$ case its diameter is rather short (poly-logarithmic in the volume), while its smallest cut is also small - as small as that of a box in the n-dimensional lattice (Cut sets polynomial in the volume).

For example, when $s = 1$, the average degree is $\theta(\log(n))$, as was shown in [8], the diameter is $\theta\left(\frac{\log(n)}{\log\log(n)}\right)$. This might lead to thinking that this graph is similar to the random graph $G(n, \log(n)/n)$. However, there exist large sets (such as the vertices $[1, ..., \frac{1}{2}n]$) that have small boundaries of order $n/logn$. This can be seen by following the argument in the proof of Theorem part (A).

6.2 electrical resistance

By the proof of Lemma 2.4 of [5], if $1 < s < 2$, there is a constant C which does not depend on N, s.t. if we pick at random two vertices of $G_{s,\beta}(N)$, then with a very high probability the effective electrical resistance between them is bounded by C. However, the maximal electrical resistance is unbounded - in fact it is easy to see that it is at least logarithmic with N.

6.3 Inverse problems

As we saw above, once the tail of the connecting probabilities is fat (but not too fat) we get a graph of poly-logarithmic diameter and super-polynomial volume growth. This brings the question whether the geometry of the underlying graph, say $\mathbb{Z}^d$ for different values of d, is disappearing and we get some universal generic geometries? To be more precise, assume you are given a sample from a super critical long-range percolation taking place on one of the two graphs $\mathbb{Z}$ or $\mathbb{Z}^2$, with some connecting probabilities, which are not given to you and without the labeling of the edges by $\mathbb{Z}$ or $\mathbb{Z}^2$. Can you a.s. tell if the sample came from long-range percolation on $\mathbb{Z}$? I.e. is the set of measures on graphs coming from considering all super critical long-range percolations on $\mathbb{Z}$ is singular with respect to that coming from $\mathbb{Z}^2$? The question could be asked for any pair of graphs, in particular can one distinguish between $\mathbb{Z}$ and the 3-regular tree T_3, or between or between $\mathbb{Z}^{d_1}$ and $\mathbb{Z}^{d_2}$? Another variant: Replace a graph by a graph roughly isometric – can one distinguish $\mathbb{Z}$ from $\mathbb{Z}^2$ but not from the triangular lattice on $\mathbb{Z}^2$?

More generally, let G be a graph, and assume you are given a sample from a super critical long-range percolation on G, with some unknown connecting probabilities and without the labeling of the vertices of the sample by their names in G. What information can be recovered on G? For instance can properties such as amenability of G, transitivity of G, number of ends in G, volume growth of G, planarity of G, and isoperimetric dimension of G, be recovered?

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